A refinement of Fontaine's map for perfectoid towers and its applications

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This talk is based on joint work with Shinnosuke Ishiro and Kazuma Shimomoto [HIS].

Perfectoid theory, introduced by P. Scholze in the early 2010s, has now become one of the most effective tools in mixed characteristic commutative algebra, following Y. André's breakthrough resolution of the direct summand conjecture. Fix a prime number p. Let A be a perfectoid ring that is ϖ-torsion free and ϖ-adically complete for a $\varpi \in A$ with $p \in \varpi^p A$. Then we obtain a perfect $\mathbb{F}_p$-algebra $A^\flat$, called the *tilt* of A, and the *Fontaine's map*

$$\sharp \colon A^\flat \to A. \tag{0.1}$$

Although (0.1) is merely multiplicative, it allows us to transfer information from A to $A^\flat$. For example, there exists a non-zero-divsisor $\varpi^\flat \in A^\flat$ corresponding to ϖ. Moreover, we have the following fundamental result in perfectoid theory:

Theorem 1 ([EHS24])**.** *If A is completely integrally closed in $A[\frac{1}{\varpi}]$, then $A^\flat$ is completely integrally closed in $A^\flat[\frac{1}{\varpi^\flat}]$.*

However, any perfectoid ring of mixed characteristic is never Noetherian, and hence does not fit into Noetherian ring theory. To overcome this difficulty, S. Ishiro, K. Nakazato, and K. Shimomoto [INS25] introduced the notion of *perfectoid towers*, a suitable class of towers of rings that approximate perfectoid rings. They also developed a *tilting theory of perfectoid towers*: given a perfectoid tower $\{R_i\}_{i\geq 0} = (R_0 \to R_1 \to R_2 \to \cdots)$ arising from some pair (R, I), the completed direct limit $\widehat{R_\infty}$ is a perfectoid ring, and one can obtain a perfectoid tower of $\mathbb{F}_p$-algebras $R_0^{s.\flat} \to R_1^{s.\flat} \to R_2^{s.\flat} \to \cdots$. In this context, we construct a refinement of Fontaine's map (0.1) for perfectoid towers:

$$\sharp^{(i)} \colon R_i^{s.\flat} \to R_i + I\widehat{R_\infty}. \tag{0.2}$$

This map, as well as (0.1), has potential applications to the theory of perfectoid towers. As one piece of evidence for this, we utilize (0.2) to obtain a *tower-theoretic analogue* of Theorem 1:

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Main Theorem A ([HIS]). *Let $\{R_i\}_{i\geq 0}$ be a perfectoid tower arising from some pair $(R, I = (f))$. Let $I^{s.\flat} = (f^{s.\flat}) \subseteq R^{s.\flat}$ be the ideal corresponding to I, and assume that $R_\infty = \bigcup_{i\geq 0} R_i$ is f-torsion free. If R_∞ is completely integrally closed in $R_\infty[\frac{1}{f}]$, then $R_n^{s.\flat}$ is completely integrally closed in $R_n^{s.\flat}[\frac{1}{f^{s.\flat}}]$ for every $n \geq 0$ so that $R_n^{s.\flat}$ is $f^{s.\flat}$-torsion free.*

On the other hand, the existence of perfectoid towers is itself a significant problem ([INS25, Ish24, IS25]). Based on F. Andreatta's argument [An06], we obtain a new construction of perfectoid towers arising from the context of almost purity:

Main Theorem B ([HIS]). *Let R be an unramified complete regular local ring of mixed characteristic with perfect residue field of characteristic $p > 0$. Let $R \to S$ be a module-finite extension of normal local domains such that $R[\frac{1}{p}] \to S[\frac{1}{p}]$ is étale. Then there exist a tower of rings $S_0 = S \to S_1 \to S_2 \to \cdots$, a rational $\varepsilon \in (0,1) \cap \mathbb{Q}$, and an integer $N \geq 0$ such that $\{S_n\}_{n\geq N}$ is a perfectoid tower arising from the pair $(S, (p^\varepsilon))$.*

As an application of Main Theorem A, we can give another proof of the following result originally obtained in [An06]:

Corollary 2. *With notation as in Main Theorem B, $S_n^{s.\flat}$ is a normal ring for every $n \geq N$.*

In this talk, we briefly review Fontaine's map and explain the construction of its refinement with applications.

References

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